To prove the Preservation of Intervals Theorem 5.3.10, we will use Theorem 2.5.1 characterizing intervals.

5.3.10 Preservation of Intervals Theorem *Let I be an interval and let $f : I \to \mathbb{R}$ be continuous on I. Then the set $f(I)$ is an interval.*

Proof. Let $\alpha, \beta \in f(I)$ with $a < \beta$; then there exist points $a, b \in I$ such that $\alpha = f(a)$ and $\beta = f(b)$. Further, it follows from Bolzano's Intermediate Value Theorem 5.3.7 that if $k \in (\alpha, \beta)$ then there exists a number $c \in I$ with $k = f(c) \in f(I)$. Therefore $[\alpha, \beta] \subseteq f(I)$, showing that $f(I)$ possesses property (1) of Theorem 2.5.1. Therefore $f(I)$ is an interval. Q.E.D.

Exercises for Section 5.3

1. Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ be a continuous function such that $f(x) > 0$ for each x in I. Prove that there exists a number $\alpha > 0$ such that $f(x) \geq \alpha$ for all $x \in I$.
2. Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be continuous functions on I. Show that the set $E := \{x \in I : f(x) = g(x)\}$ has the property that if $(x_n) \subseteq E$ and $x_n \to x_0$, then $x_0 \in E$.
3. Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ be a continuous function on I such that for each x in I there exists y in I such that $|f(y)| \leq \frac{1}{2}|f(x)|$. Prove there exists a point c in I such that $f(c) = 0$.
4. Show that every polynomial of odd degree with real coefficients has at least one real root.
5. Show that the polynomial $p(x) := x^4 + 7x^3 - 9$ has at least two real roots. Use a calculator to locate these roots to within two decimal places.
6. Let f be continuous on the interval [0, 1] to $\mathbb{R}$ and such that $f(0) = f(1)$. Prove that there exists a point c in $[0, \frac{1}{2}]$ such that $f(c) = f\left(c + \frac{1}{2}\right)$. [*Hint*: Consider $g(x) = f(x) - f\left(x + \frac{1}{2}\right)$.] Conclude that there are, at any time, antipodal points on the earth's equator that have the same temperature.
7. Show that the equation $x = \cos x$ has a solution in the interval $[0, \pi/2]$. Use the Bisection Method and a calculator to find an approximate solution of this equation, with error less than 10^{-3}.
8. Show that the function $f(x) := 2 \ln x + \sqrt{x} - 2$ has root in the interval [1, 2], Use the Bisection Method and a calculator to find the root with error less than 10^{-2}.
9. (a) The function $f(x) := (x - 1)(x - 2)(x - 3)(x - 4)(x - 5)$ has five roots in the interval [0, 7]. If the Bisection Method is applied on this interval, which of the roots is located?
   (b) Same question for $g(x) := (x - 2)(x - 3)(x - 4)(x - 5)(x - 6)$ on the interval [0, 7].
10. If the Bisection Method is used on an interval of length 1 to find p_n with error $|p_n - c| < 10^{-5}$, determine the least value of n that will assure this accuracy.
11. Let $I := [a, b]$, let $f : I \to \mathbb{R}$ be continuous on I, and assume that $f(a) < 0, f(b) > 0$. Let $W := \{x \in I : f(x) < 0\}$, and let $w := \sup W$. Prove that $f(w) = 0$. (This provides an alternative proof of Theorem 5.3.5.)
12. Let $I := [0, \pi/2]$ and let $f : I \to \mathbb{R}$ be defined by $f(x) := \sup\{x^2, \cos x\}$ for $x \in I$. Show there exists an absolute minimum point $x_0 \in I$ for f on I. Show that x_0 is a solution to the equation $\cos x = x^2$.
13. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $\lim_{x \to -\infty} f = 0$ and $\lim_{x \to \infty} f = 0$. Prove that f is bounded on $\mathbb{R}$ and attains either a maximum or minimum on $\mathbb{R}$. Give an example to show that both a maximum and a minimum need not be attained.
14. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and let $\beta \in \mathbb{R}$. Show that if $x_0 \in \mathbb{R}$ is such that $f(x_0) < \beta$, then there exists a δ-neighborhood U of x_0 such that $f(x) < \beta$ for all $x \in U$.

15. Examine which open [respectively, closed] intervals are mapped by $f(x) := x^2$ for $x \in \mathbb{R}$ onto open [respectively, closed] intervals.
16. Examine the mapping of open [respectively, closed] intervals under the functions $g(x) := 1/(x^2 + 1)$ and $h(x) := x^3$ for $x \in \mathbb{R}$.
17. If $f : [0, 1] \to \mathbb{R}$ is continuous and has only rational [respectively, irrational] values, must f be constant? Prove your assertion.
18. Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ be a (not necessarily continuous) function with the property that for every $x \in I$, the function f is bounded on a neighborhood $V_{\delta_x}(x)$ of x (in the sense of Definition 4.2.1). Prove that f is bounded on I.
19. Let $J := (a, b)$ and let $g : J \to \mathbb{R}$ be a continuous function with the property that for every $x \in J$, the function g is bounded on a neighborhood $V_{\delta_x}(x)$ of x. Show by example that g is not necessarily bounded on J.

Section 5.4 Uniform Continuity

Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Definition 5.1.1 states that the following statements are equivalent:

(i) f is continuous at every point $u \in A$;

(ii) given $\varepsilon > 0$ and $u \in A$, there is a $\delta(\varepsilon, u) > 0$ such that for all x such that $x \in A$ and $|x - u| < \delta(\varepsilon, u)$, then $|f(x) - f(u)| < \varepsilon$.

The point we wish to emphasize here is that δ depends, in general, on *both* $\varepsilon > 0$ and $u \in A$. The fact that δ depends on u is a reflection of the fact that the function f may change its values rapidly near certain points and slowly near other points. [For example, consider $f(x) := \sin(1/x)$ for $x > 0$; see Figure 4.1.3.]

Now it often happens that the function f is such that the number δ can be chosen to be independent of the point $u \in A$ and to depend only on ε. For example, if $f(x) := 2x$ for all $x \in \mathbb{R}$, then

$$|f(x) - f(u)| = 2|x - u|,$$

and so we can choose $\delta(\varepsilon, u) := \varepsilon/2$ for all $\varepsilon > 0$ and all $u \in \mathbb{R}$. (Why?)

On the other hand if $g(x) := 1/x$ for $x \in A := \{x \in \mathbb{R} : x > 0\}$, then

$$g(x) - g(u) = \frac{u - x}{ux}. \tag{1}$$

If $u \in A$ is given and if we take

$$\delta(\varepsilon, u) := \inf\left\{\tfrac{1}{2}u, \tfrac{1}{2}u^2\varepsilon\right\}, \tag{2}$$

then if $|x - u| < \delta(\varepsilon, u)$, we have $|x - u| < \frac{1}{2}u$ so that $\frac{1}{2}u < x < \frac{3}{2}u$, whence it follows that $1/x < 2/u$. Thus, if $|x - u| < \frac{1}{2}u$, the equality (1) yields the inequality

$$|g(x) - g(u)| \le (2/u^2)|x - u|. \tag{3}$$

Consequently, if $|x - u| < \delta(\varepsilon, u)$, then (2) and (3) imply that

$$|g(x) - g(u)| < (2/u^2)\left(\tfrac{1}{2}u^2\varepsilon\right) = \varepsilon.$$

We have seen that the selection of $\delta(\varepsilon, u)$ by the formula (2) "works" in the sense that it enables us to give a value of δ that will ensure that $|g(x) - g(u)| < \varepsilon$ when $|x - u| < \delta$ and

We shall close this section by stating the important theorem of Weierstrass concerning the approximation of continuous functions by polynomial functions. As would be expected, in order to obtain an approximation within an arbitrarily preassigned $\varepsilon > 0$, we must be prepared to use polynomials of arbitrarily high degree.

5.4.14 Weierstrass Approximation Theorem *Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be a continuous function. If $\varepsilon > 0$ is given, then there exists a polynomial function p_ε such that $|f(x) - p_\varepsilon(x)| < \varepsilon$ for all $x \in I$.*

There are a number of proofs of this result. Unfortunately, all of them are rather intricate, or employ results that are not yet at our disposal. (A proof can be found in Bartle, ERA, pp. 169–172, which is listed in the References.)

Exercises for Section 5.4

1. Show that the function $f(x) := 1/x$ is uniformly continuous on the set $A := [a, \infty)$, where a is a positive constant.
2. Show that the function $f(x) := 1/x^2$ is uniformly continuous on $A := [1, \infty)$, but that it is not uniformly continuous on $B := (0, \infty)$.
3. Use the Nonuniform Continuity Criterion 5.4.2 to show that the following functions are not uniformly continuous on the given sets.
   (a) $f(x) := x^2, \quad A := [0, \infty)$.
   (b) $g(x) := \sin(1/x), \quad B := (0, \infty)$.
4. Show that the function $f(x) := 1/(1 + x^2)$ for $x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.
5. Show that if f and g are uniformly continuous on a subset A of $\mathbb{R}$, then $f + g$ is uniformly continuous on A.
6. Show that if f and g are uniformly continuous on $A \subseteq \mathbb{R}$ and if they are *both* bounded on A, then their product fg is uniformly continuous on A.
7. If $f(x) := x$ and $g(x) := \sin x$, show that both f and g are uniformly continuous on $\mathbb{R}$, but that their product fg is not uniformly continuous on $\mathbb{R}$.
8. Prove that if f and g are each uniformly continuous on $\mathbb{R}$, then the composite function $f \circ g$ is uniformly continuous on $\mathbb{R}$.
9. If f is uniformly continuous on $A \subseteq \mathbb{R}$, and $|f(x)| \geq k > 0$ for all $x \in A$, show that $1/f$ is uniformly continuous on A.
10. Prove that if f is uniformly continuous on a bounded subset A of $\mathbb{R}$, then f is bounded on A.
11. If $g(x) := \sqrt{x}$ for $x \in [0, 1]$, show that there does not exist a constant K such that $|g(x)| \leq K|x|$ for all $x \in [0, 1]$. Conclude that the uniformly continuous g is not a Lipschitz function on $[0, 1]$.
12. Show that if f is continuous on $[0, \infty)$ and uniformly continuous on $[a, \infty)$ for some positive constant a, then f is uniformly continuous on $[0, \infty)$.
13. Let $A \subseteq \mathbb{R}$ and suppose that $f : A \to \mathbb{R}$ has the following property: for each $\varepsilon > 0$ there exists a function $g_\varepsilon : A \to \mathbb{R}$ such that g_ε is uniformly continuous on A and $|f(x) - g_\varepsilon(x)| < \varepsilon$ for all $x \in A$. Prove that f is uniformly continuous on A.
14. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **periodic** on $\mathbb{R}$ if there exists a number $p > 0$ such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. Prove that a continuous periodic function on $\mathbb{R}$ is bounded and uniformly continuous on $\mathbb{R}$.

15. Let f and g be Lipschitz functions on A.
    (a) Show that the sum $f + g$ is also a Lipschitz function on A.
    (b) Show that if f and g are bounded on A, then the product fg is a Lipschitz function on A.
    (c) Give an example of a Lipschitz function f on $[0, \infty)$ such that its square f^2 is *not* a Lipschitz function.

16. A function is called *absolutely continuous* on an interval I if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any pair-wise disjoint subintervals $[x_k, y_k]$, $k = 1, 2, \ldots, n$, of I such that $\sum |x_k - y_k| < \delta$ we have $\sum |f(x_k) - f(y_k)| < \varepsilon$. Show that if f satisfies a Lipschitz condition on I, then f is absolutely continuous on I.

Section 5.5 Continuity and Gauges†

We will now introduce some concepts that will be used later—especially in Chapters 7 and 10 on integration theory. However, we wish to introduce the notion of a "gauge" now because of its connection with the study of continuous functions. We first define the notion of a tagged partition of an interval.

5.5.1 Definition A **partition** of an interval $I := [a, b]$ is a collection $\mathcal{P} = \{I_1, \ldots, I_n\}$ of non-overlapping closed intervals whose union is $[a, b]$. We ordinarily denote the intervals by $I_i := [x_{i-1}, x_i]$, where

$$a = x_0 < \cdots < x_{i-1} < x_i < \cdots < x_n = b.$$

The points x_i $(i = 0, \ldots, n)$ are called the **partition points** of $\mathcal{P}$. If a point t_i has been chosen from each interval I_i, for $i = 1, \ldots, n$, then the points t_i are called the **tags** and the set of ordered pairs

$$\dot{\mathcal{P}} = \{(I_1, t_1), \ldots, (I_n, t_n)\}$$

is called a **tagged partition** of I. (The dot signifies that the partition is tagged.)

The "fineness" of a partition $\mathcal{P}$ refers to the lengths of the subintervals in $\mathcal{P}$. Instead of requiring that all subintervals have length less than some specific quantity, it is often useful to allow varying degrees of fineness for different subintervals I_i in $\mathcal{P}$. This is accomplished by the use of a "gauge," which we now define.

5.5.2 Definition A **gauge** on I is a strictly positive function defined on I. If δ is a gauge on I, then a (tagged) partition $\dot{\mathcal{P}}$ is said to be δ-**fine** if

$$(1) \qquad t_i \in I_i \subseteq [t_i - \delta(t_i),\ t_i + \delta(t_i)] \quad \text{for} \quad i = 1, \ldots, n.$$

We note that the notion of δ-fineness requires that the partition be tagged, so we do not need to say "tagged partition" in this case.

A gauge δ on an interval I assigns an interval $[t - \delta(t),\ t + \delta(t)]$ to each point $t \in I$. The δ-fineness of a partition $\dot{\mathcal{P}}$ requires that each subinterval I_i of $\dot{\mathcal{P}}$ is contained in the interval determined by the gauge δ and the tag t_i for that subinterval. This is indicated by the inclusions in (1); see Figure 5.5.1. Note that the length of the subintervals is also controlled by the gauge and the tags; the next lemma reflects that control.

†This section can be omitted on a first reading of this chapter.